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ABSTRACT

Five sets of activities for students are included in this document. Each is designed for use in junior high and secondary school mathematics instruction. The first "Note" concerns magic squares in which the numbers in every row, column, and diagonal add up to the same sum. An etching by Albrecht Durer is presented, with four questions followed by activities on rotations and reflections with magic squares, the Lo-Shu magic square, building a magic square, diabolic squares, and a magic cube. Challenge problems are also included. The other "Notes," structured similarly with a variety of activities, concern: Random Walks; Probability: Quantifying Change; Tessellations; Patterns in Geometry; and Correlation: What Makes a Perfect Pair? Solutions are included. (MNS)

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NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
STUDENT math notes
JANUARY 1985

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Magic Squares

Albrecht Dürer (1471–1528) was a famous German artist. He lived during the Renaissance and was one of the first to use the geometry of perspective in his work. His etching titled **Melancolia** contains the 4×4 magic square exactly as shown here. Notice how cleverly he incorporated the date 1514.

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16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

The numbers in every row, column, and diagonal of a magic square have the same **magic sum.**

1. What is the magic sum for this magic square?
2. How many rows, columns, and diagonals can you find with this magic sum?
3. Find five 2×2 squares in the magic square with four numbers that have the same sum as the magic sum.
4. Now try to find four numbers, each from a different row and column, that also have the same sum as the magic sum. There are two solutions besides the two diagonals.

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Rotations and Reflections

Four different rotations are possible for the magic square in figure 1A. Rotate this square clockwise 0° (A), 90° (B), 180° (C), and 270° (D). Complete the magic squares B, C, and D.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

A

4			16
1			

B

			16

C

D

Fig. 1

If you **reflect** each of the four magic squares in figure 1 about its vertical axis, another magic square is formed for each. (See figure 2.) Complete these reflections in the four squares in figure 2.

13			16
1			4

A

16			4
			1

B

16			

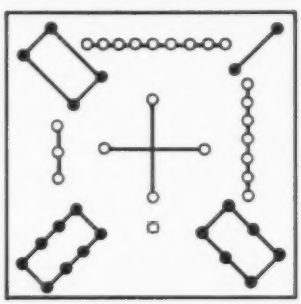
C

D

Fig. 2

The Lo-Shu Magic Square

The oldest known magic square appeared on an ancient Chinese tablet that dates back to 2200 B.C. It is called the **Lo-Shu magic square**. Legend has it that the square was first drawn on a tortoise shell given to the emperor **Yu.** Translate the 3 × 3 Lo-Shu magic square, pictured at right, into base-ten Hindu-Arabic numerals.



Lo-Shu Magic Square

Hindu-Arabic numerals

Magic squares can be transformed into other magic squares by adding, multiplying, or dividing all the numbers by any constant value. Transform the Lo-Shu magic square by first adding 2 to each cell and then dividing by 8. Show the results in the form of simplified fractions. Then verify that the new square is, indeed, another magic square.

Building a 5 × 5 Magic Square

Here is a method for constructing a 5 × 5 magic square using the numbers 1 through 25.

Start with 1 in the middle of the top row. (See figure 1.) Move one square up and to the right for each successive number.

1. If that position is outside the square, place the next number either five squares down from it (as with 2) or five squares to the left of it (as with 4).
2. If that position is already filled, place the number directly below the last square just filled (as with 6).

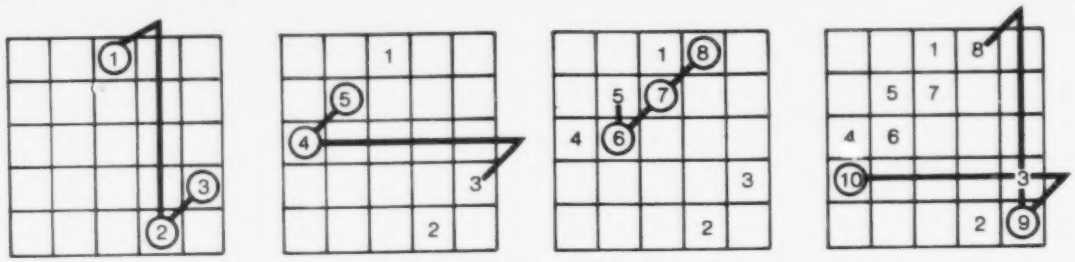


Fig. 1

Follow the placement of the numbers 1 through 10 using these rules. Then complete the magic square. The number 16 will go under the number 15. Check to see that the result is a magic square by finding the sums of the five rows, five columns, and two diagonals. What is the magic sum?

Diabolic Squares

Every magic square can be transformed by rotations and reflections into other magic squares. **Diabolic magic squares** also allow a row to be transferred from top to bottom, bottom to top, left to right, and right to left.

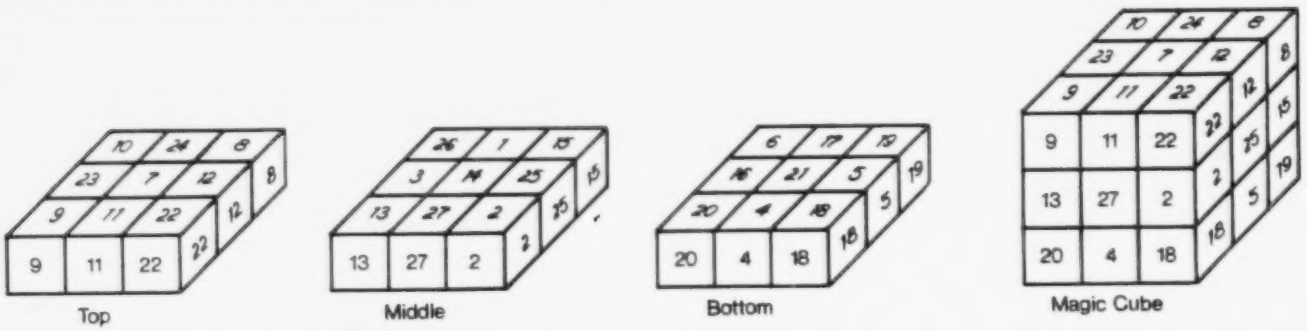
A 4 × 4 diabolic magic square has been repeated over and over again to form the number grid shown below.

7	12	1	14	7	12	1	14	7	12	1	14	7	12	1	14	7	12	1	14
2	13	8	11	2	13	8	11	2	13	8	11	2	13	8	11	2	13	8	11
16	3	10	5	16	3	10	5	16	3	10	5	16	3	10	5	16	3	10	5
9	6	15	4	9	6	15	4	9	6	15	4	9	6	15	4	9	6	15	4
7	12	1	14	7	12	1	14	7	12	1	14	7	12	1	14	7	12	1	14
2	13	8	11	2	13	8	11	2	13	8	11	2	13	8	11	2	13	8	11
16	3	10	5	16	3	10	5	16	3	10	5	16	3	10	5	16	3	10	5
9	6	15	4	9	6	15	4	9	6	15	4	9	6	15	4	9	6	15	4

1. Draw a 4 × 4 square around **any** 16 numbers in the grid. Check to see if the square you chose is also a magic square.
2. See how many different 4 × 4 squares you can find on the grid this way. Is each one a magic square?
3. Suppose you take one of those 4 × 4 magic squares that you found, curl it up top to bottom, and then join it end to end, as shown, to form a doughnut shape called a **torus.** Any four numbers that are connected side to side or corner to corner will have the same sum. Try making a model of a **magic torus.**

A Magic Cube

Imagine these three 3 × 3 arrays of cubes stacked together to form the 3 × 3 × 3 magic cube shown.



There are 39 ways that three numbered cubes joined in a row, face-to-face, edge-to-edge, or vertex-to-vertex have the same sum, 42. List all those that you can find.

Did you know that . . .

- There are 880 different ways to enter the numbers 1 through 16 in a 4 × 4 magic square, not counting rotations and reflections.
- Benjamin Franklin was very interested in magic squares. He invented an 8 × 8 magic square with a magic sum of 260 not only for its 18 rows, columns, and diagonals but in many other ways as well.
- Connecting the center points of the 16 squares in Dürer's magic square in numerical order forms an interesting symmetric design. Try it.
- The sequence of 18 digits below that repeat in the decimal expansion for 1/19 can be used to form an 18 × 18 magic square.

  $$1/19 = .\overline{052631578947368421}\ldots$$

  For the 18 rows, list in numerical order, as shown in the box, the appropriate 18 successive repeating digits.

05263157 . . . . .
10526315 . . . . .
15789473 . . . . .
21052631 . . . . .
26315789 . . . . .
. . . . .
. . . . .

Bet you can't . . .

- Construct a 3 × 3 magic square with the magic sum of 2.
- Construct a 5 × 5 magic square using the integers −12 through 12.
- Construct a 7 × 7 magic square using the numbers 1 through 49 by following the rules given for constructing a 5 × 5 magic square.
- Use variables to construct a 3 × 3 magic square if the center square is n.
- Complete this 3 × 3 magic square with repeating decimals if the magic sum is 3.

$.4\overline{5}$		
	1	
	$.\overline{63}$	

Solutions. *Page 1:* **1)** 34; **2)** 10; **3)** In the four corners and the center; **4)** 3, 8, 9, 14 and 2, 5, 12, 15.
Page 3: The magic sum for the 5 × 5 square is 65.
Each of the 16 numbers can be used in a given corner to form a diabolic square.

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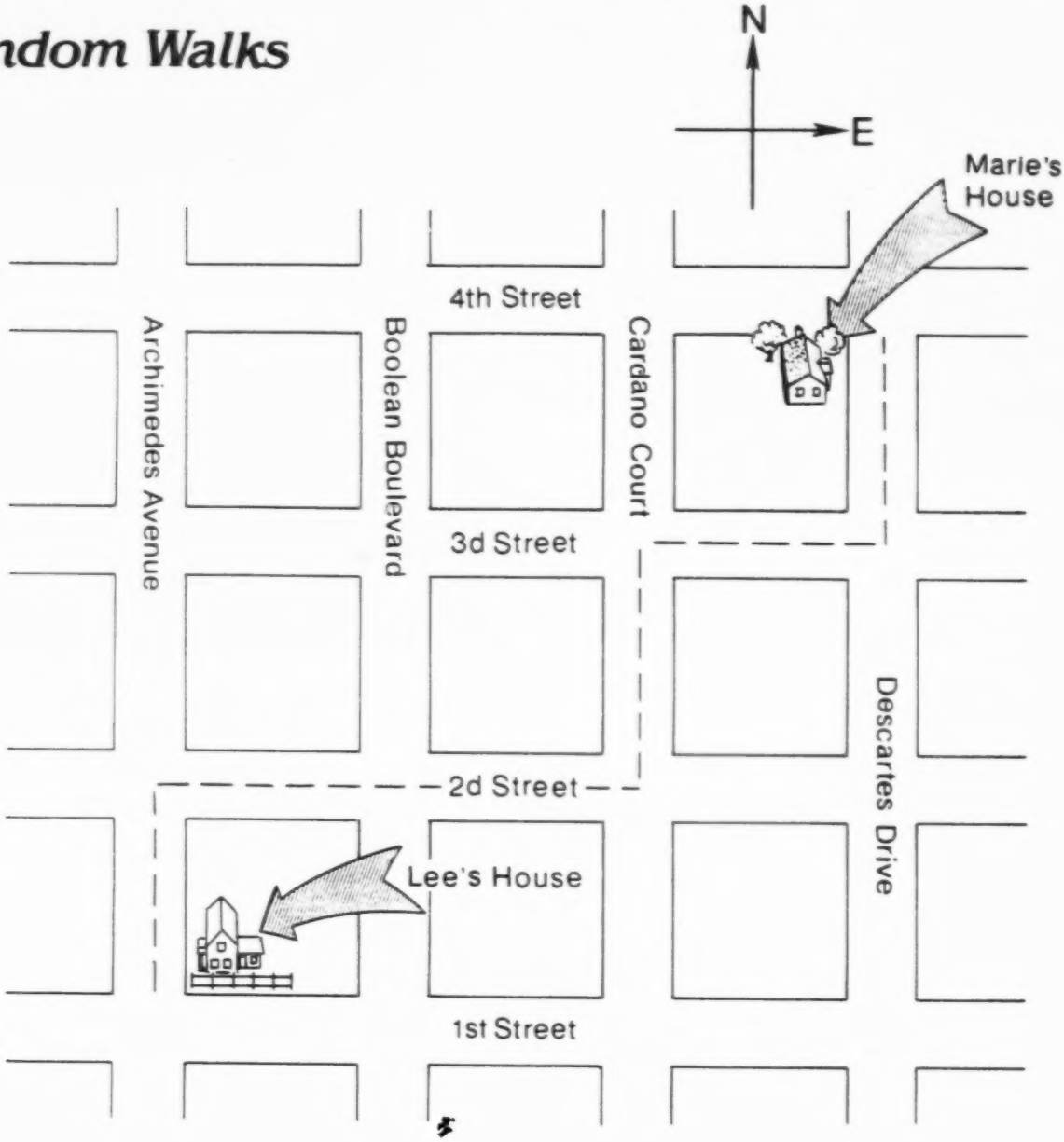
STUDENT math notes
MARCH 1985

Random Walks

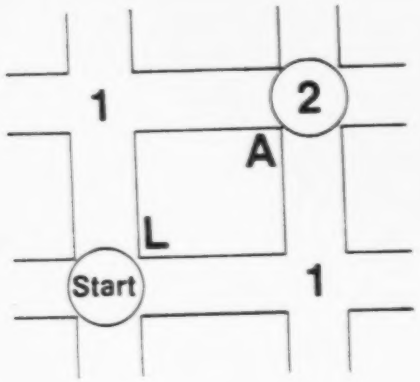


How many different ways are there for Lee to get from his house to Marie's house if he can only travel **east** or **north** on the streets shown?

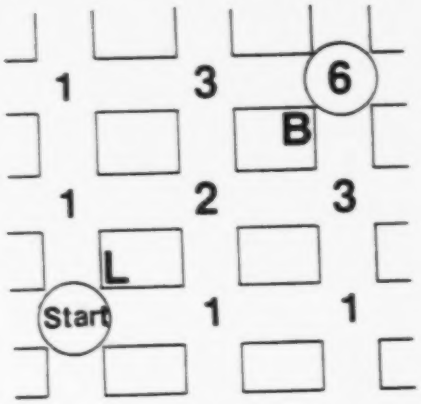
Trace and record each path you can find. For example, the path traced on the map can be recorded as A-2-2-C-3-D. Use a letter or a number to represent the street traveled for **each** of the six blocks in a given path.

Looking at a Simpler Case

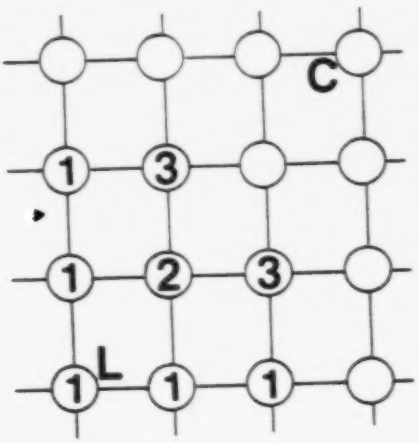
Let's look at some simpler cases. To keep track of the possible paths, put a number in each intersection to represent the number of ways you can get to that intersection traveling only **east** or **north**.



How many ways are there from **L** to **A**? _____



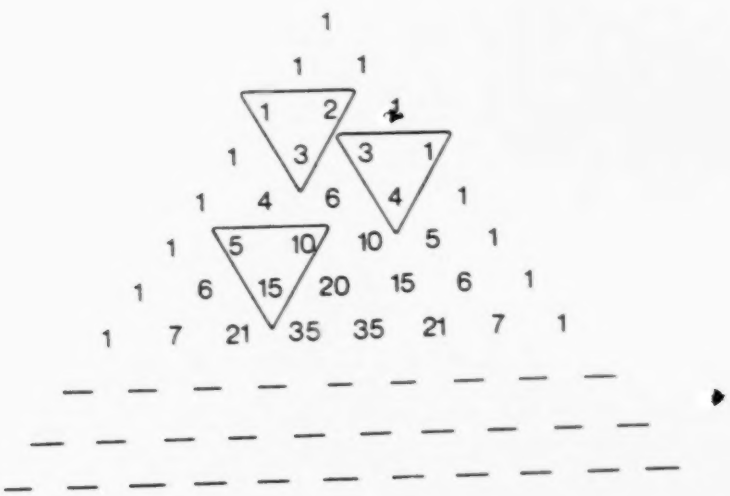
How many ways are there from **L** to **B**? _____



Complete the numbers in the grid to find the number of ways from **L** to **C**. _____

Can you find a pattern to the numbers in the intersections of these maps?

Pascal's Triangle



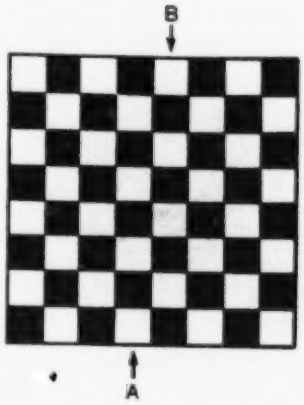
The pattern for the numbers in the intersections of the **random walk** problems is found in a triangular array known as **Pascal's triangle**. Two sides of the triangle always contain 1's. The other values are found by adding the two numbers directly above them as shown in the three small triangular regions marked.

Complete the next three rows in Pascal's triangle.

Extending Pascal's Triangle

On the checkerboard at the right, how many different ways are there to move a checker from position **A** to position **B** so it can be made a king? Try to use Pascal's triangle to help you find the answer. You may want to look at a few simpler cases first by starting with **B** closer to **A.**

What constraints did the edges of the checkerboard create as you tried to apply Pascal's triangle?



Word Search

Start at the center. At each succeeding move, move right, left, up, or down. How many different paths can you find to spell the word MATH?

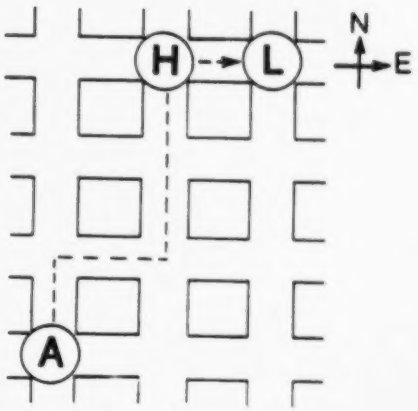
How many ways can you find to spell the word PASCAL? Each move from the center must be to the right, left, up, or down.

Did you discover how Pascal's triangle can be used to help solve these word search problems?

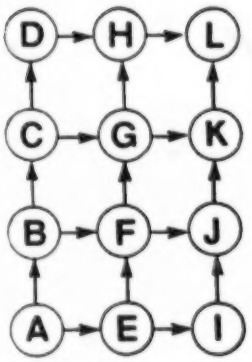
Probability of a Random Walk

Recently, for the sake of variety, Lee decided that at each intersection **where he had a choice** he would flip a "fair coin" to determine whether he would go north (heads) or east (tails). If Lee is walking from intersection **A** to meet Marie at intersection **L**, what is the probability that he will pass through intersection **H**? Remember that the probability of heads or tails on a fair coin is 1/2.

One possible path is shown. It resulted from the four tosses HTHH. Note that there is no choice on the fifth move, so no toss is needed.



The Probabilities Simplified



Let's look at a simpler case first. If Lee starts at **A** on his way to **L**, then the probability that he gets to **B** is 1/2, since he flips the fair coin at **A**. Likewise, by symmetry, the probability that he gets to **E** is also 1/2.

The probability (P) that he gets to **F** is the sum of the probabilities of coming from **B** and from **E**. For example, P(F from B) = 1/2 • P(B) = 1/2 • 1/2, or 1/4. Likewise, P(F from E) is also 1/4. Thus P(F), the probability that he gets to F from B or E, is 1/4 + 1/4, or 1/2.

In a similar fashion, P(C) = P(I) = 1/4. Since there is **no choice** of direction in going from I to J, P(J from I) = 1 • P(I) = 1 • 1/4, or 1/4.

What is P(J from F)? _____

What is P(J)? _____

Now complete the missing probabilities for the grid. If you have computed the probabilities correctly, with **L** the final destination, P(L) should equal 1.

$P(D) = __$	$P(H) = __$	$P(L) = __$
$P(C) = 1/4$	$P(G) = __$	$P(K) = __$
$P(B) = 1/2$	$P(F) = 1/2$	$P(J) = __$
$P(A) = 1$	$P(E) = 1/2$	$P(I) = 1/4$

Did you know that . . .

- Pascal's triangle also contains the coefficients for the expansion of the binomial $(x + y)$ to any integral exponent. Use the numbers in the appropriate row of Pascal's triangle to expand $(x + y)^n$ for $n = 0, 1, 2, 3, \ldots$

$$
\begin{aligned}
(x + y)^0 &= 1 \\
(x + y)^1 &= 1x^1 + 1y^1 \\
(x + y)^2 &= 1x^2 + 2xy + 1y^2 \\
(x + y)^3 &= 1x^3 + 3x^2y^1 + 3x^1y^2 + 1y^3 \\
(x + y)^4 &= 1x^4 + 4x^3y^1 + 6x^2y^2 + 4x^1y^3 + 1y^4
\end{aligned}
$$

- Blaise Pascal, for whom Pascal's triangle is named, lived from 1623 to 1662. Pascal was known as a brilliant mathematician as early as age 12. The notion of mathematical probability emerged from a debate Pascal had with Pierre de Fermat over a game of chance. As a result, Pascal is often referred to as the father of probability.

Bet you can't . . .

- Expand $(x + y)^{10}$.
- Find the number of ways to get from the lower left corner of a checkerboard to the upper right corner by only moving up or to the right.

Solutions. *Page 1:* There are 20 different paths from Lee's house to Marie's house.
Page 3: Checkerboard problem: 35.
MATH word search: 28.
The probability of a random walk from A to L passing through H: 5/16 (Be careful not to toss the coin at **every** intersection. No path from A to L requires five tosses. The sequence ends when either three heads or two tails are reached.)

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MAY 1985

Probability: Quantifying Chance

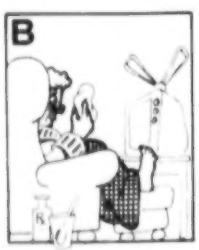
If chance will have me king, why, chance may crown me,
Without my stir.

Shakespeare's *Macbeth*

Chance affects us all, just as it did Macbeth. Few of the activities of your life are exempt from it. You talk about the chance that your team will win the game, the chance that you will pass a test, the chance that you'll win a school election, and so forth.

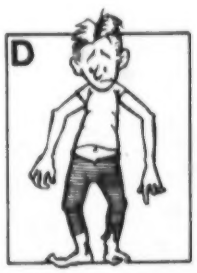
Think about the degree of chance that each of the events listed below will happen, and then assign them a position on the number line. You have five choices: 0 represents an impossibility, and 1 represents a certainty. The fractions represent the points in between. Place each event's letter on the number line at the point you think best describes the chance that that event will occur. The first one is marked for you.

It will rain in your city this year.



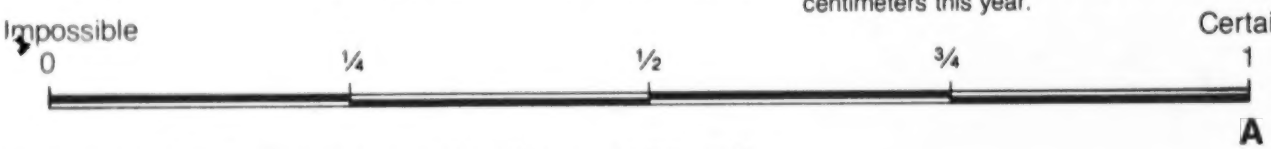
You will miss a day of school this year.

You will earn an A on your next math test.



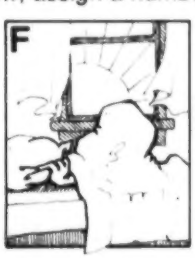
Your height will increase at least 3 centimeters this year.

You will hear a telephone ring today.



Not all students will choose the same numbers that you did. Why not?

Now, assign a number from 0 through 1 to the chance that each of these events will occur.

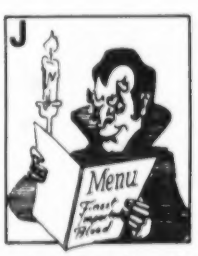


The sun will rise tomorrow.

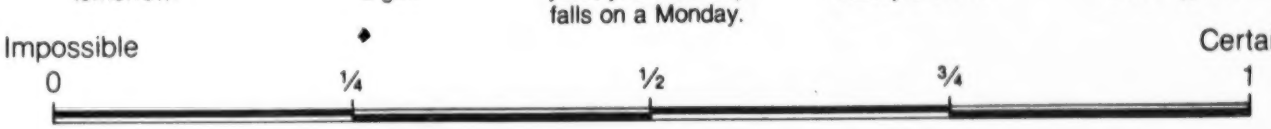
A newborn baby is a girl.

In an unspecified year, your birthday falls on a Monday.

You have seen Halley's comet.



A person has the blood type O.

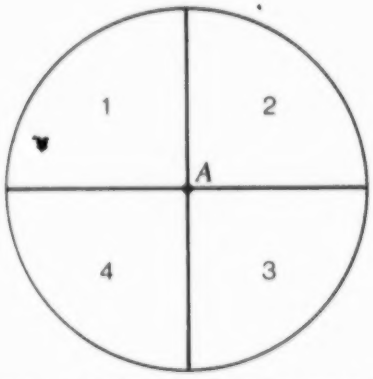


Most students will choose the same numbers that you did. Do you know why?

Round and Round It Goes; Where It Stops, Probability Knows

Although we cannot control chance, we can "mathematize," or quantify, chance by associating numbers with the likelihood that an event will happen. This is called **computing probabilities.** The word **probability** is used interchangeably with the word **chance.**

Let's consider some chances. Lay a paper clip on circle *A*, holding one end of it at the center of the circle with the tip of your pencil. Use your other hand to spin the paper clip. Which section of the circle does it stop on? Spin 20 times and mark the result of each spin in the table below.



Section	Tallies	Total
1		
2		
3		
4		
		20

Which number did the clip stop on most frequently? On the basis of this activity, which number has the greatest probability of being stopped on? _____ Which has the least? _____

Looking again at the table of outcomes for your 20 spins, write the fraction that represents the total number of 1's you spun divided by the total number of spins. _____ This fraction is the probability that a "1" will occur on a spin. Use your table of spins to compute the probability of spinning each of the other three numbers:

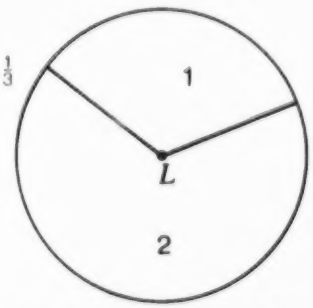
2 _____ **3** _____ **4** _____

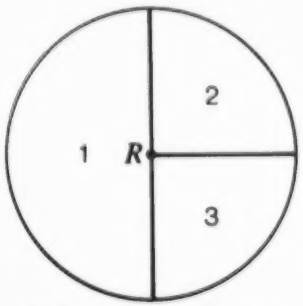
You probably could have predicted the outcome of your 20 spins. Since the four sectors of the circle are of equal area, the paper clip is equally likely to stop on each of them. There is 1 chance out of 4, or a probability of ¼, that it will stop on the 1. What is your prediction that it will stop on the 2? _____ On the 3? _____ On the 4? _____

Compare the probabilities you predicted with those you computed on the basis of your spinning experience. They are probably different. As you can see, figuring probabilities in advance, or theoretically, does not dictate what will actually happen.

Look at these two circles. Circle *L* is divided into two regions, one of which is twice as large as the other. Circle *R* is divided into three regions, two of which are equal to each other and half that of the third.

Circle L: Section 1 is $\frac{1}{3}$ of the circle.



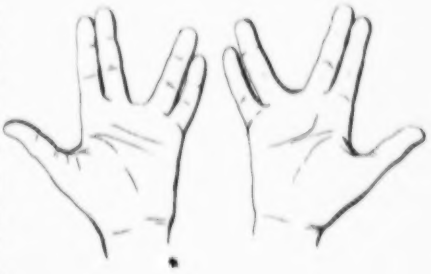


Circle R: Section 1 is $\frac{1}{2}$ of circle; the others are $\frac{1}{4}$ each.

Alternating between circles *L* and *R*, make 24 pairs of spins on the two circles. After each pair of spins, add the two results. For instance, if you spin a 2 on *L* and a 3 on *R*, the sum is 5. Keep a record of how many sums of 2, 3, 4, and 5 you obtain. Theoretically, a sum of 3 is most likely. Why? _____ On your 24 pairs of spins, which sum is most common? _____ Of the remaining sums (2, 4, and 5), which do you think would be most common?

What If Probability Doesn't Know?

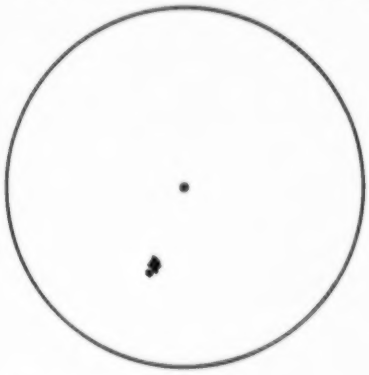
As you've seen from your spinning activities, what mathematics predicts in a probabilistic situation and what actually happens are usually somewhat different. Try as we might, we just cannot always correctly predict chance. Sometimes, there's simply no way to compute a theoretical probability, or there are several different theoretical probabilities for the same situation.



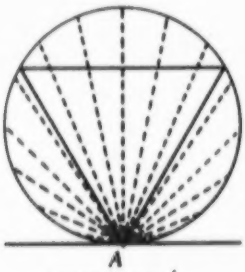
Inspired by watching the television show *Star Trek,* a student devised a test for left-handedness. The idea came from First Officer Spock's familiar V-shaped gesture. Spread the fingers of each of your hands apart so as to form as large a V as possible with two fingers forming each segment of the V. Now observe which hand has its fingers stretched farther apart—the left or the right. According to the test, if the right hand has its fingers stretched farther apart, the person is left-handed. The test was performed on 345 left-handed people, and 289 were correctly identified as being "lefties." According to these data, what is the probability that a person predicted by this test to be left-handed is actually left-handed? _____ Administer the V-sign test to your friends and see if you think your data indicate that the test is reliable. There's no theoretical probability to guide you. You're on your own.

Sometimes more than one theoretical probability can be computed for the same problem. In the nineteenth century, the French mathematician Joseph Bertrand worked on such a problem:

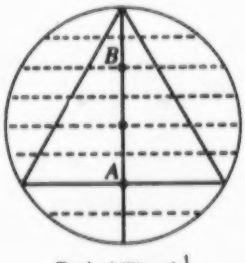
In the circle, draw a chord of any length you like. (Draw it before you read any further.) Now, use your compass and straightedge to construct an equilateral triangle that is inscribed in a circle. Compare the length of the random chord you drew to the length of the side of the triangle. Which is longer? Thirty mathematics students did this, and 23 of them (about 75%) drew chords that had a length shorter than a side of the triangle. For them, chance appears to favor random chords shorter than the side of the triangle.



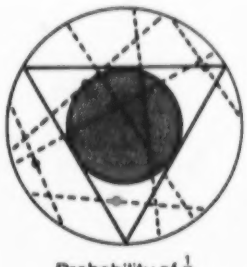
Computing the theoretical probability in this problem is interesting in that there are three ways to do the computation, depending on how the problem is interpreted. All three techniques yield different answers: ⅓, ½, and ¼. The geometrical diagrams below are the basis for computing each of the different probabilities. This paradox of three answers for this problem stems from there being no precise definition of how to draw a random chord. See if you can figure out how each of these probabilities was obtained.



Probability of $\frac{1}{3}$



Probability of $\frac{1}{2}$



Probability of $\frac{1}{4}$

Probabilities and Odds

Can you convert **probabilities** to **odds?** Both words are different ways of saying the same thing. The probability of an event happening is the **ratio** of the number of **successful** outcomes to the number of **possible** outcomes. The probability of tossing tails on one throw of a coin is ½. The odds in favor of an event happening is the ratio of the number of **successful** outcomes to the number of **unsuccessful** outcomes. The odds in favor of tossing tails on one throw of a coin are 1:1. What are the odds in favor of tossing a "6" on one throw of a die?

Bet you can't . . .

- Count the number of each color (brown, yellow, orange, green, and tan) of M&M's Plain Chocolate Candies in a bag and use the results to determine the probabilities of drawing each of the five colors out of the bag without looking.
- Find the probability of tossing tails on the 100th flip of a coin that has turned up heads on each of the preceding 99 flips.

Bet you didn't know that . . .

- According to officials of Mars, makers of M&M's Plain Chocolate Candies, there are 40% brown, 20% yellow, 20% orange, 10% green, and 10% tan candies in each bag. The company used consumer preference tests to determine which color assortment pleased the greatest number of people.
- M&M's Peanut Chocolate Candies come in only four colors, with an average blend of 25% of each color.
- Throwing nails on a floor made of narrow boards is an example of a probability technique that was important in designing shielding for atomic reactors.
- The probability that a bridge hand will contain **4 aces** is approximately 1/379. (Bridge is a card game in which each hand contains 13 cards.)
- The odds in favor of your drawing the **jack of diamonds** from a deck of 52 shuffled playing cards is 1:51.
- In January 1978, four Iowa women playing bridge claimed they were all dealt "perfect" bridge hands (each player having all 13 cards of the same suit). The odds in favor of such an occurrence are 1 to more than 2 octillion. (An octillion is 1 followed by 27 zeros: 1 000 000 000 000 000 000 000 000 000.)
- The study of probability started in Italy and France about 400 years ago in an effort to devise strategies for gambling in dice games.
- Gregor Mendel of Austria (1822–1884) was one of the first people to apply probability theory to science when he crossbred plants to study genetics.

Hints and Help

- For more about the V-sign test (p. 3), see *Current Science,* 20 February 1974.
- For more about Bertrand's probability paradox, see Martin Gardner's *Second Scientific American Book of Mathematical Puzzles and Diversions* (Simon & Schuster, 1961, pp. 223–26).

Solutions. *Page 2:* Regarding circles *L* and *R*:
The probability of a **sum of 3** is $\frac{1}{3} \cdot \frac{1}{4} + \frac{2}{3} \cdot \frac{1}{2}$, or $\frac{5}{12}$.
Sum of 4: $\frac{1}{3} \cdot \frac{1}{4} + \frac{2}{3} \cdot \frac{1}{4}$, or $\frac{3}{12}$
Sum of 2: $\frac{2}{12}$
Sum of 5: Same as sum of 2 ($\frac{2}{12}$)

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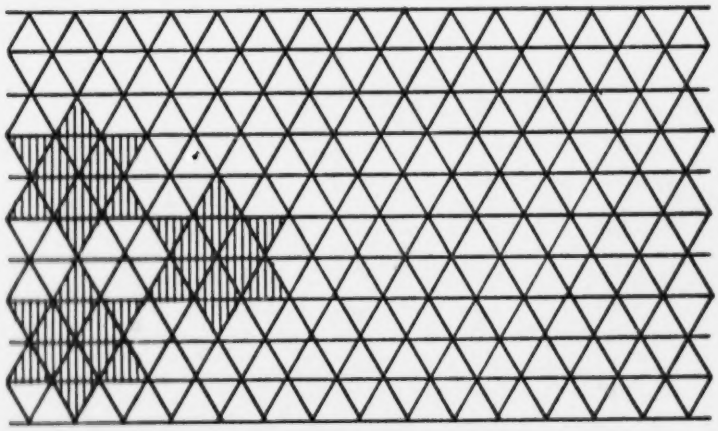
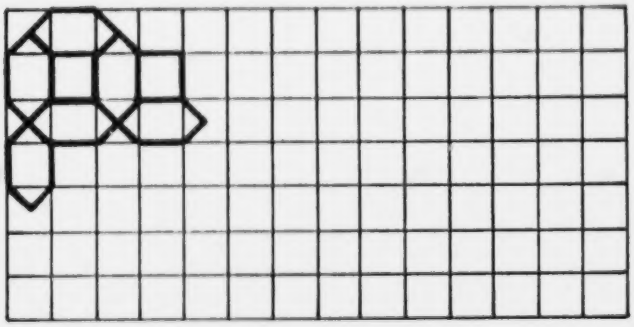
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
STUDENT math notes
SEPTEMBER 1985

Tessellations: Patterns in Geometry

To **tessellate a plane** means to completely cover a surface with a pattern of shapes with no gaps and no overlapping. Many designs are **geometric tessellations**.

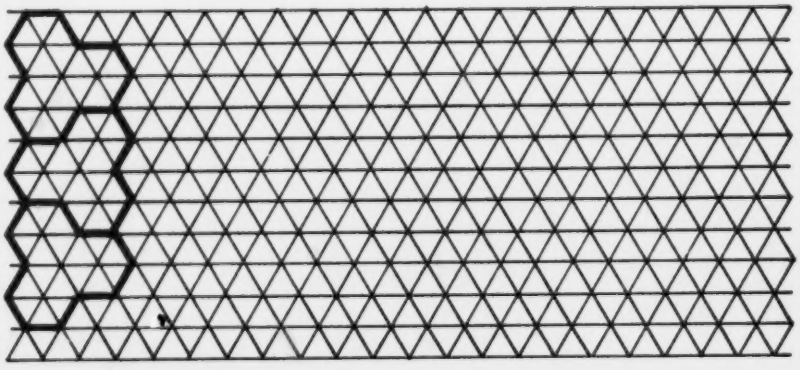


Use this triangular grid to complete the star-and-hexagon tessellation that has been started.



Use the lines of the square grid to help you extend the tessellation of squares and hexagons. Can you see the overlapping octagons?

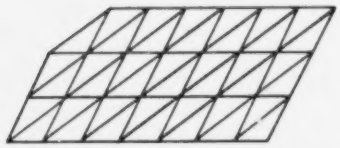
Complete a tessellation of these shapes across the page. Color your finished tessellation with three colors. Do not have two shapes of the same color touch each other.

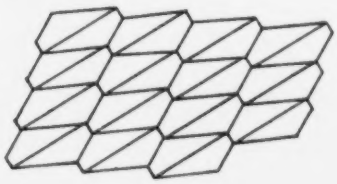


The editors wish to thank Dale Seymour, P.O. Box 10888, Palo Alto, CA 94303, for writing this issue of *NCTM Student Math Notes*.

Tessellations of Triangles and Quadrilaterals

Below are some examples of triangles and quadrilaterals that tessellate a plane. Notice that in these tessellations the polygons are joined edge to edge and the arrangement of angles about each vertex is the same.





Do *all* triangles tessellate the plane? Let's try to find out.

1. Cut six or more *congruent* triangles out of paper.
2. Label the angles on the triangles so that all angle 1's are congruent, all angle 2's are congruent, and all angle 3's are congruent.
3. Place the triangles on a table or desk top to form a tessellation.
4. Do you have angles 1, 2, and 3 together at a common vertex? What appears to be the sum of angles 1, 2, and 3? Do you think this is true of all triangles?
5. Do your triangles tessellate in a pattern like the ones shown in figure 1?

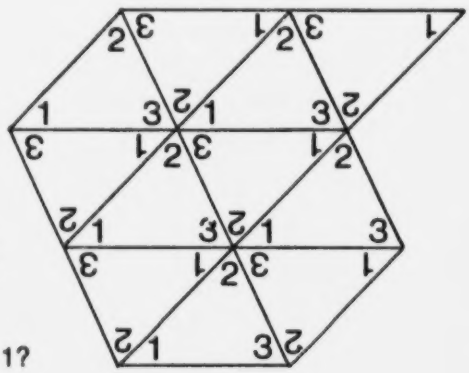
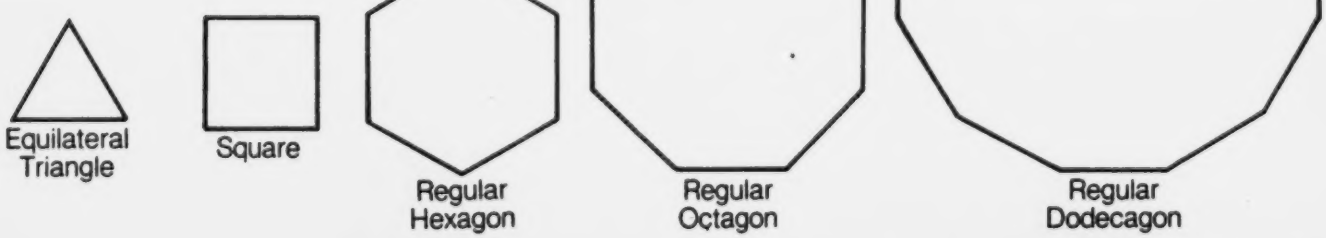


Fig. 1

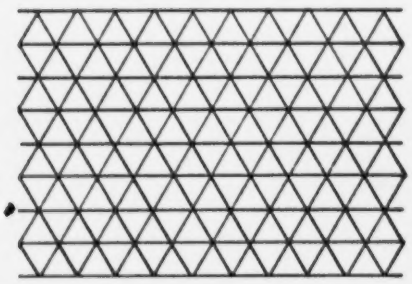
6. Any triangle will tessellate the plane. How about quadrilaterals? Repeat the experiment above with six or more congruent quadrilaterals (four-sided polygons). Do your quadrilaterals tessellate?

Here are some **regular polygons**. Each has all sides congruent and all angles congruent. In these illustrations, the sides of the different regular polygons are also all the same length. The polygons can serve as patterns for the tessellation activity described on the next page.

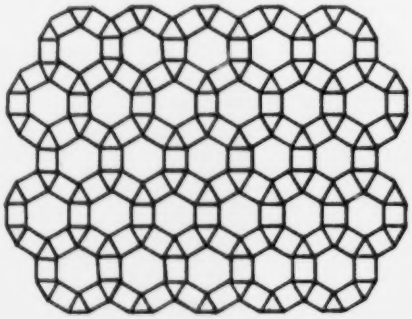


Regular Tessellations and Semiregular Tessellations

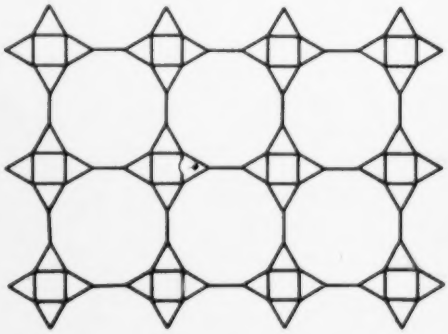
Regular tessellations use *one* type of regular polygon. **Semiregular** tessellations are formed by a combination of two or more types of regular polygons. Each vertex of the semiregular tessellation must be formed by the same arrangement of regular polygons. Polygons must be placed edge to edge. Tessellations are named by the shapes that meet at a vertex: "4.3.4.6" means "square, triangle, square, hexagon."



Regular:
Every vertex is 3.3.3.3.3.3.



Semiregular:
Every vertex is 4.3.4.6.



Neither regular nor semiregular:
Some vertices are 4.3.12.3, and others are 12.3.12.

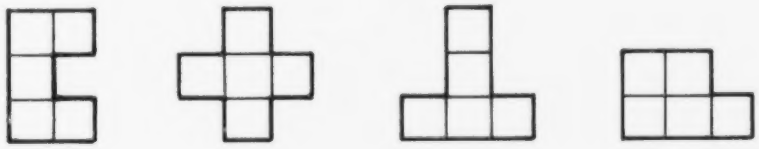
Cut out some regular polygons using the patterns given on page 2. Use them to help you complete the chart below to find all eight semiregular tessellations. Part of the chart is already completed. Each column heading in the table gives the number of sides and the measure of each angle for the regular polygon used. Be sure the arrangements for 2 and 3 and for 5 and 6 are different.

Angles in Each Regular Polygon

	3(60°)	4(90°)	6(120°)	8(135°)	12(150°)	Tessellation
1.	1	2	1	0	0	4.3.4.6 (shown above)
2.	3	2	0	0	0	3.3.4.3.4
3.	3	2	0	0	0	
4.	0		0		0	
5.		0		0	0	
6.		0		0	0	
7.	0			0		
8.		0	0	0		

Tessellating Pentominoes

There are twelve different arrangements of five congruent squares that meet at their edges. These are called pentominoes. Four of them are shown below.



1. Use a sheet of graph paper and try to draw the other eight pentominoes.
2. Experiment to determine which pentominoes tessellate.

Bet you can't. . .

- Show how some letters of the alphabet can be drawn so that they tessellate.
- Discover a pentagon that tessellates.
- Make a tessellation of a combination of regular polygons that is different from the eight semiregular tessellations. (All vertex points don't have to be formed by identical arrangements of regular polygons.)
- Construct each of the eight semiregular tessellations with a compass and straightedge.

Bet you didn't know that. . .

- *Tesselae* means "tile" in Latin.
- Geometric mosaics were used as decorations as early as 4000 B.C.
- Numerous polygonal tessellations were pictured in Johannes Kepler's book, *Harmonice Mundi*, published in 1619.
- The eleven regular and semiregular tessellations are often called Archimedean tilings.

Hints and Help

- Try filling all 360 degrees about a point in a plane when searching for the semiregular tessellation combinations (p. 3).
- In your search to find all eight semiregular tessellations of regular polygons (p. 3), don't overlook the possibility of **different arrangements** with triangles and squares.
- All twelve pentominoes (p. 4) will tessellate without being reflected (flipped). Try drawing these shapes. Think **patterns** and tessellating with them.

Solutions. *(Page 3):*
3) 3.3.3.4.4 4) 4.8.8 5) 3.3.3.3.6 6) 3.6.3.6 7) 4.6.12 8) 3.12.12

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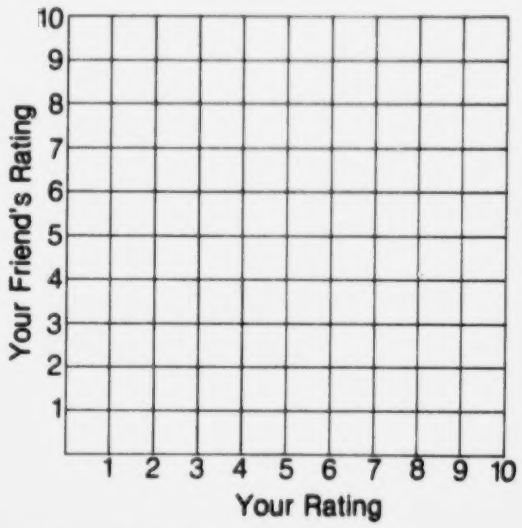
STUDENT math notes
NOVEMBER 1985

Correlation: What Makes a Perfect Pair?

How alike are you and your best friend? If both of you had to rate a list of television shows or movies, how closely would your ratings match? Let's find out how you and your friend or a classmate rate the shows in the list below. Use **1** for the **favorite** and **10** for the **least favorite.** Now pick a friend and each of you rate the list of shows.

TV Show	Your Ratings	Your Friend's Ratings
A. "A-Team"	_____	_____
B. "Cosby Show"	_____	_____
C. "Knight Rider"	_____	_____
D. "Hill Street Blues"	_____	_____
E. "60 Minutes"	_____	_____
F. "Fall Guy	_____	_____
G. "Carol Burnett and Friends"	_____	_____
H. "Family Ties"	_____	_____
I. "Dallas"	_____	_____
J. "Airwolf"	_____	_____

How closely do your ratings compare? One way to find out is to make a picture, or **scatter plot.** On the grid to the right, plot **ordered pairs** consisting of the two ratings for each show. Label the points with the letters of the corresponding shows in the listing above. Point A will correspond to "A-Team" and be placed according to the ordered pair (your rating, your friend's rating). For example, if you rated "A-Team" fourth and your friend rated it second, then place the letter A on point (4,2). Plot these pairs on the grid.



How closely does your rating match that of your friend? What would the scatter plot of a perfect match look like? What would the scatter plot of exactly opposite ratings look like?

The editors wish to thank Gail Burrill, Whitnall High School, 5000 S. 116 St., Greenfield, WI 53228, for writing this issue of *NCTM Student Math Notes.*

Rank Correlation of Ordered Pairs

Three pairs of friends rated the 10 shows listed on page 1 as shown below.

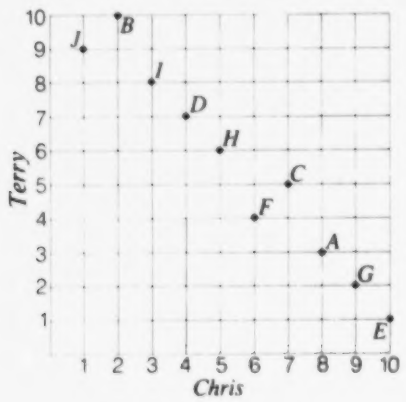


Figure 1

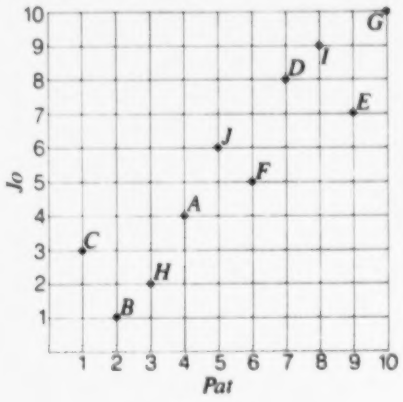


Figure 2

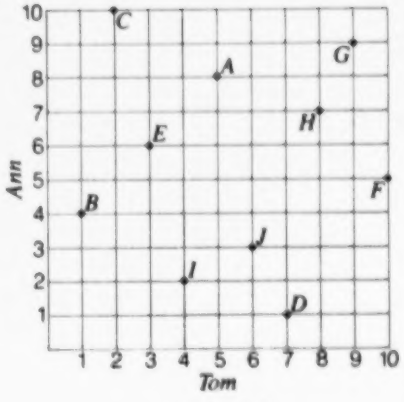


Figure 3

1. Do Chris and Terry agree or disagree on their ratings of "60 Minutes"? _____
2. Which shows do Pat and Jo rate the same? _____
3. Which scatter plot indicates general agreement on ratings? _____
4. Which scatter plot indicates general disagreement on ratings? _____
5. Describe the comparison of ratings made by Tom and Ann. _____

The picture formed by a scatter plot gives an impression of how the two ratings match or correlate with each other. It shows how closely the points form a line, but the picture is not very definite or easy to describe. A numerical description called the **correlation coefficient** gives a clearer idea. This numerical description is represented by the letter r. The type of correlation used in a comparison of preferences, such as this matching of television shows, is called **rank correlation.**

Formula for Rank Correlation

Here is the formula to find the rank correlation:

$$r = 1 - \frac{6\Sigma d^2}{n(n^2-1)}$$

Here are the definitions of the parts of the formula:

- r - correlation coefficient
- d - the difference between the two ratings
- Σ - the Greek letter **sigma**, a mathematical shorthand for **sum**
- n - the number of items that were ranked
- 6 - a constant in the formula

As an example of how this works, let's use the data from figure 1 above. For the given ratings, r is found as shown to the right.

Point	Chris	Terry	d	d^2
A	8	3	5	25
B	2	10	−8	64
C	7	5	2	4
D	4	7	−3	9
E	10	1	9	81
F	6	4	2	4
G	9	2	7	49
H	5	6	−1	1
I	3	8	−5	25
J	1	9	−8	64
				326

Insert the appropriate numbers in the formula, and solve.

$$r = 1 - \frac{6(326)}{10(99)}$$
$$= 1 - 1.98$$
$$= -0.98$$

Positive and Negative Correlation

The correlation coefficient r is always less than or equal to **1** and greater than or equal to **−1.** A strong positive correlation can be 0.75 to 1, depending on the definition of strong. A strong negative correlation is −0.75 to −1 and means the ratings for the items were almost opposite.

1. From the information above, how would you interpret the correlation coefficient of −0.98 from the example on page 2?

Now compute the rank correlation coefficient using the data from figures 2 and 3 on page 2 and from the scatter plot you developed on page 1.

From Figure 2

Point	Pat	Jo	d	d^2
A				
B				
C				
D				
E				
F				
G				
H				
I				
J				

$r =$ ____

A

From Figure 3

Point	Tom	Ann	d	d^2
A				
B				
C				
D				
E				
F				
G				
H				
I				
J				

$r =$ ____

B

You and Your Friend

Point	You	Friend	d	d^2
A				
B				
C				
D				
E				
F				
G				
H				
I				
J				

$r =$ ____

C

2. How would you interpret the correlation coefficient for these tables? _____
3. If two persons' ratings were exactly the same, what would Σd^2 be? _____
   How would this influence the value of r? _____
4. What would you expect the value of r to be if there were no relationship in the rankings? _____

Cause and Effect

The most common error in working with correlation is to confuse a strong correlation with cause and effect. Correlation means **association,** not necessarily **causation.** If there is a high positive correlation between the size of your feet and the size of your hands, it does not mean that large feet cause large hands. Large feet and hands usually belong to a tall person; one does not cause the other. If there is a strong correlation between time spent on homework and scores on a unit test, however, it is likely that the time spent on homework helped determine the test scores.

Research indicates that the following have a strong correlation. What do you think about the cause-and-effect relation?

1. Smoking and lung cancer
2. Age and the frequency of automobile accidents
3. Alcohol intake by expectant mothers and birth abnormalities of their infants
4. Sugar consumed and increased hyperactivity
5. Cholesterol consumed and heart disease
6. Exercise and good health
7. Marijuana use and genetic defects in offspring
8. Use of vitamin C and number of colds
9. Amount of education and yearly income

Further Activities on Correlation

Bet you can't . . .

- Find two rankings of the same five items to make $r = 0$.
- Find all the different values possible for r when two people rank the same four items.
- Predict the correlation as positive, negative, or close to 0 for reading ability and shoe size.
- Predict the correlation as positive, negative, or close to 0 for the number of points a player makes in a basketball game and the number of fouls he earns.

Bet you didn't know that . . .

- The study of correlation in statistics began in the late 1800s with Sir Francis Galton, who was a cousin of Charles Darwin. Sir Francis wanted to predict the adult heights of sons from the heights of their fathers.
- October 18 is National Statistics Day in Japan.
- "Some men use statistics as a tired man uses a lamppost, for support rather than illumination."—Anonymous

Hints and Helps: This computer program will calculate r for any number of ranked items. It was written by Scott Trent, a student at Whitnall High School, Greenfield, Wisconsin.

```
 1 REM   RANK CORRELATION COEFFICIENT FOR A SET OF N RANKINGS
 2 REM
 3 REM   VARIABLES USED
 4 REM   A$,B$          : STRING USER INPUT
 5 REM   A(N),B(N)      : CONVERTED NUMERIC USER INPUT (ORDERED PAIR)
 6 REM   N              : NUMBER OF ORDERED PAIRS (INTEGER)
 7 REM   SUM            : SUM OF DIFFERENCES OF ORDERED PAIRS SQUARED
 8 REM   R              : CALCULATED R VALUE
 9 REM   I              : FOR/NEXT COUNTER
10 REM
50 DIM A(500),B(500)                              :REM  SET UP ARRAYS FOR STORING PAIRS
100 PRINT "Rank Correlation Coefficient for a set of N rankings."
105 PRINT
110 PRINT "Enter ordered pairs, type, 'END,END' when finished"
120 PRINT
130 INPUT A$,B$                                   :REM  GET USER INPUT
140 IF A$="END" OR B$="END" THEN 200              :REM  IF AT END THEN BRANCH TO LINE 200
145 IF A$="'END" OR B$="END'" THEN 200            :REM  TEST FOR AMBIGUOUS INPUT
150 A(N)=VAL(A$):B(N)=VAL(B$)                     :REM  CONVERT STRING INPUT TO NUMERIC
160 SUM=SUM+(A(N)-B(N))^2                         :REM  ADD UP THE DIFFERENCES SQUARED
170 N=N+1                                         :REM  KEEP TRACK OF THE NUMBER OF PAIRS
180 GOTO 130                                      :REM  LOOP TO LINE 130 FOR ANOTHER PAIR
200 R=1-(6*SUM)/(N*(N^2-1))                       :REM  COMPUTE R USING STANDARD EQUATION
210 R=INT(R*10000)/10000                          :REM  ROUND R TO 4 PLACES
220 PRINT                                         :REM  LEAVE A SPACE BEFORE PAUSE
230 INPUT "Press return to list data";A$          :REM  PAUSE
240 PRINT                                         :REM  LEAVE A SPACE BEFORE DATA
250 FOR I=0 TO N-1                                :REM  INITIATE LOOP
260 PRINT "(";A(I);",";B(I);")"                   :REM  DISPLAY ORDERED PAIR
270 NEXT I                                        :REM  CONCLUSION OF LOOP
280 PRINT                                         :REM  LEAVE A SPACE BEFORE ANSWER
290 PRINT "The rank correlation coefficient is ";R :REM  PUT RESULT ON SCREEN
999 END
```

Solutions. *Page 2:* **1)** Disagree **2)** "A-Team," and "Carol Burnett" **3)** Fig. 2 **4)** Fig. 1 **5)** No strong agreement or disagreement
Page 3: **1)** Strong negative correlation, hence strong disagreement. For fig. 1, $r = -0.98$; for fig. 2, $r = 0.92$ (rounded to two decimal places) **2)** (A) Strong agreement; (B) neither strong agreement nor strong disagreement **3)** 0; r would be exactly 1 **4)** $r = 0$
Page 4: **Some possible rankings** of five items that make $r = 0$: 1, 2, 3, 4, 5 and 4, 3, 2, 1, 5; 1, 2, 3, 4, 5 and 1, 5, 4, 3, 2. **All possible r values** when two people rank the same four items are 1, .8, .6, .4, .2, 0, −.2, −.4, −.6, −.8, −1

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